Thermodynamic properties of a trapped relativistic Bose gas with pair production

2008 J. Phys. A: Math. Theor. 41 285002

(http://iopscience.iop.org/1751-8121/41/28/285002)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 285002 (8pp) doi:10.1088/1751-8113/41/28/285002

Thermodynamic properties of a trapped relativistic Bose gas with pair production

Guozhen Su, Liwei Chen and Jincan Chen

Department of Physics and Institute of Theoretical Physics and Astrophysics, Xiamen University, Xiamen 361005, People's Republic of China

E-mail: gzsu@xmu.edu.cn

Received 26 January 2008, in final form 29 April 2008
Published 19 June 2008
Online at stacks.iop.org/JPhysA/41/285002

Abstract

An ideal gas of relativistic bosons trapped in a D-dimensional power-law potential is explored, where the possibility of particle–antiparticle pair production is taken into account. Expressions for several important parameters, such as the critical temperature of Bose–Einstein condensation (BEC), ground-state fraction and heat capacity, are analytically derived and used to discuss the relevant properties of the system. An important parameter χ is introduced to mark the extent of the relativistic effects on the characteristics of BEC. It is found that the correction of the critical temperature due to the relativistic effects is considerable for the system with a large parameter χ. It is more important to show that the pair production may lead to several novel characteristics for a relativistic Bose gas, such as the high temperature BEC, the rapid increase of the heat capacity at high temperatures and the disappearance of the anomaly of the heat capacity at the critical temperature in the ultrarelativistic limit.

PACS numbers: 05.30.Jp, 03.75.Hh, 03.30.+p

1. Introduction

In most works dealing with the behavior of degenerate Bose gases, nonrelativistic energy dispersion is adopted since the relativistic effects are often negligible. However, there are some systems in the universe for which the relativistic effects are considerable. For example, in the final stage of heavy ion collisions [1, 2], a large number of high-energy bosonic hadrons produced in the process can be approximately described as an ideal gas of bosons, and the energy dispersion should be treated relativistically for low-mass particles whose kinetic energies are comparable with their rest energies, i.e., $k_B T = \mathrm{O}(mc^2)$, where k_B is the Boltzmann constant, T is the temperature, m is the rest mass of a particle and c is the velocity of light.

The properties of relativistic Bose gases have been investigated by several authors [3–14]. The early works mainly concentrated on the corrections to the properties of Bose gases due to the relativistic effects [3–6], where the total number of particles N was assumed to be a conserved quantity. More complete treatment of relativistic Bose systems was given by using the techniques of quantum field theory, in which the possibility of particle–antiparticle pair production was considered and a more general conserved quantity $Q = N - \bar{N}$, which may be generically referred to as the 'charge' number [7], was introduced in place of N [7–14], where $\bar{N}$ is the number of antiparticles.

In [15] the authors of this paper studied the Bose–Einstein condensation of a trapped relativistic Bose gas, in which we mainly concentrated on the corrections resulting from the relativistic energy spectrum and did not consider the effects of particle–antiparticle pair production. At high temperatures such that $k_BT = \mathrm{O}(mc^2)$, the probability of pair production cannot be ignored and the inclusion of pair production becomes necessary. The present paper will continue the work in [15] and focus on how pair production affects the behavior of a relativistic Bose gas in the presence of an external potential.

2. General expressions of the important thermodynamic quantities

We consider an ideal relativistic Bose gas trapped in a D-dimensional generic power-law potential with single-particle energy

$$\varepsilon(p, x_1, x_2, \ldots, x_D) = \sqrt{p^2c^2 + m^2c^4} + \sum_{k=1}^{D} \varepsilon_k \left|\frac{x_k}{L_k}\right|^{t_k}, \tag{1}$$

where p and x_k are, respectively, the momentum and kth component of the coordinate of a particle, t_k, ε_k and L_k ($k = 1, 2, \ldots, D$) are all positive constants that mark the shape and strength of the external potential. The power-law potential adopted here represents a class of traps to restrict the particles and may take different forms if the different parameters t_k, ε_k and L_k are chosen. For example, it is reduced to a rigid confining box if $t_k \to \infty$ and a 'harmonic' trap if $t_k = 2$.

According to the quantum field theory, the total number of 'charges' for the ideal Bose system with pair production is given by [7]

$$Q = N - \bar{N} = \sum_{\varepsilon} \left\{ \frac{1}{\exp[\beta(\varepsilon - \mu)] - 1} - (\mu \to -\mu) \right\}, \tag{2}$$

where $\beta = 1/k_BT$ and μ is the chemical potential of the system. It should be noted that the numbers of the particles and antiparticles in any state should be positive. This requires that the absolute value of the chemical potential should not exceed the lowest energy of the single-particle states, i.e., $|\mu| \leqslant mc^2$. Moreover, if it is further assumed that at the initial time of the system, $Q > 0$, one has $0 < \mu \leqslant mc^2$.

For the system satisfying the thermodynamic limit [16], the sum over the states may be replaced by the integral over the phase space. According to equations (1) and (2), the total number of 'charges' can be expressed as

$$\begin{aligned} Q &= Q_0 + \frac{1}{h^D} \left\{ \int \frac{\prod_{k=1}^{D} \mathrm{d}x_k \, \mathrm{d}p_k}{\exp\left[\beta\left(\sqrt{p^2c^2 + m^2c^4} + \sum_{k=1}^{D} \varepsilon_i |x_k/L_k|^{t_k}\right) - \mu\right] - 1} - (\mu \to -\mu) \right\} \\ &= Q_0 + \frac{\Omega}{\lambda_{nr}^D} \left(\frac{8u}{\pi}\right)^{1/2} \sum_{j=1}^{\infty} \frac{\sinh(j\beta\mu)}{j^{D'+\eta-1}} \mathrm{K}_{D'}(ju), \end{aligned} \tag{3}$$

where $u = \beta mc^2$, $D' = (D+1)/2$, $\eta = \sum_{k=1}^{D} 1/t_k$, h is the Planck constant, $\lambda_{nr} = h/\sqrt{2\pi mk_BT}$ is the nonrelativistic thermal wavelength,

$$\Omega = \prod_{k=1}^{D} \frac{(2L_k)\Gamma(1/t_k+1)}{(\beta\varepsilon_k)^{1/t_k}} \tag{4}$$

is the 'pseudovolume' introduced in [17],

$$\mathrm{K}_\nu(x) = \frac{\sqrt{\pi}}{\Gamma(\nu+1/2)}\left(\frac{x}{2}\right)^\nu \int_0^\infty \exp(-x\cosh\theta)\sinh^{2\nu}\theta\,\mathrm{d}\theta \tag{5}$$

is the modified Bessel function, and

$$Q_0 = \frac{1}{\exp[(u-\beta\mu)]-1} - (\mu \to -\mu) \tag{6}$$

is the ground-state occupation number of 'charges'.

When $\mu \to mc^2$ and the ground-state occupation number of 'charges' is still macroscopically negligible, i.e., $Q_0 = 0$, BEC begins to occur in the system. According to equation (3), the critical temperature T_C is determined by

$$Q = \frac{\Omega_C}{\lambda_{C,nr}^D}\left(\frac{8u_C}{\pi}\right)^{1/2}\sum_{j=1}^{\infty}\frac{\sinh(ju_C)}{j^{D'+\eta-1}}\mathrm{K}_{D'}(ju_C), \tag{7}$$

where $\lambda_{C,nr} = h/\sqrt{2\pi mk_BT_C}$, $u_C = \beta_C mc^2$, $\beta_C = 1/k_BT_C$, and

$$\Omega_C = \prod_{k=1}^{D}\frac{(2L_k)\Gamma(1/t_k+1)}{(\beta_C\varepsilon_k)^{1/t_k}}. \tag{8}$$

From equations (3) and (7), one can obtain the ground-state fraction of 'charges' at $T < T_C$ as

$$\frac{Q_0}{Q} = 1 - \left(\frac{T}{T_C}\right)^{D'+\eta-1}\frac{\sum_{j=1}^{\infty}\sinh(ju)\mathrm{K}_{D'}(ju)/j^{D'+\eta-1}}{\sum_{j=1}^{\infty}\sinh(ju_C)\mathrm{K}_{D'}(ju_C)/j^{D'+\eta-1}}. \tag{9}$$

Similarly, the total energy of the system is given by

$$E = \sum_{\varepsilon}\left\{\frac{\varepsilon}{\exp[\beta(\varepsilon-\mu)]-1} + (\mu\to-\mu)\right\}. \tag{10}$$

Under the thermodynamic limit, equation (10) can be expressed as

$$E = E_0 + \frac{\Omega k_BT}{\lambda_{nr}^D}\left(\frac{8u}{\pi}\right)^{1/2}\sum_{j=1}^{\infty}\frac{\cosh(j\beta\mu)}{j^{D'+\eta}}[(\eta-1)\mathrm{K}_{D'}(ju) + ju\mathrm{K}_{D'+1}(ju)], \tag{11}$$

where

$$E_0 = \frac{mc^2}{\exp[(u-\beta\mu)]-1} + (\mu\to-\mu) \approx \frac{mc^2}{\exp[(u-\beta\mu)]-1} - (\mu\to-\mu) = Q_0mc^2, \tag{12}$$

because the term $(\mu \to -\mu)$ in the above equation is macroscopically negligible.

According to equations (3), (11) and (12), one can further calculate the heat capacity at the given number of 'charges' and external potential. When $T > T_C$, $Q_0 = 0$, $E_0 = 0$, and the heat capacity can be derived as

$$C_{T>T_C} = \frac{\mathrm{d}E}{\mathrm{d}T} = \left(\frac{\partial E}{\partial T}\right)_\mu + \left(\frac{\partial E}{\partial \mu}\right)_T \frac{\mathrm{d}\mu}{\mathrm{d}T}$$

$$= \frac{\Omega k_B}{\lambda_{nr}^D}\left(\frac{8u}{\pi}\right)^{1/2}\left(\sum_{j=1}^{\infty}\frac{\cosh(j\beta\mu)}{j^{D'+\eta}}[\eta(\eta-1)\mathrm{K}_{D'}(ju) + (2\eta-3)ju\mathrm{K}_{D'+1}(ju)\right.$$

$$+ j^2u^2\mathrm{K}_{D'+2}(ju)] - \left\{\sum_{j=1}^{\infty}\frac{\sinh(j\beta\mu)}{j^{D'+\eta-1}}[(\eta-1)\mathrm{K}_{D'}(ju) + ju\mathrm{K}_{D'+1}(ju)]\right\}^2 \Bigg/$$

$$\left.\sum_{j=1}^{\infty}\frac{\cosh(j\beta\mu)}{j^{D'+\eta-2}}\mathrm{K}_{D'}(ju)\right), \tag{13}$$

where the property of the modified Bessel function

$$x\frac{\mathrm{d}\mathrm{K}_\nu(x)}{\mathrm{d}x} = \nu\mathrm{K}_\nu(x) - x\mathrm{K}_{\nu+1}(x) \tag{14}$$

is employed. When $T \leqslant T_C$, $\mu = mc^2$ and one can obtain

$$C_{T\leqslant T_C} = \frac{\mathrm{d}E}{\mathrm{d}T}$$

$$= \frac{\Omega k_B}{\lambda_{nr}^D}\left(\frac{8u}{\pi}\right)^{1/2}\left(\sum_{j=1}^{\infty}\frac{\cosh(ju)}{j^{D'+\eta}}\{[\eta(\eta-1) + j^2u^2]\mathrm{K}_{D'}(ju) + (2\eta-3)ju\mathrm{K}_{D'+1}(ju)\right.$$

$$\left.+ j^2u^2\mathrm{K}_{D'+2}(ju)\} - 2u\sum_{j=1}^{\infty}\frac{\sinh(ju)}{j^{D'+\eta-1}}[(\eta-1)\mathrm{K}_{D'}(ju) + ju\mathrm{K}_{D'+1}(ju)]\right). \tag{15}$$

By using equations (13) and (15), the jump of the heat capacity between $T \to T_C^-$ and $T \to T_C^+$ is found to be

$$\Delta C = C_{T\to T_C^-} - C_{T\to T_C^+}$$

$$= \frac{\Omega_C k_B}{\lambda_{C,nr}^D}\left(\frac{8u_C}{\pi}\right)^{1/2}\left\{\sum_{j=1}^{\infty}\frac{\sinh(ju_C)}{j^{D'+\eta-1}}[(\eta-1)\mathrm{K}_{D'}(ju_C) + ju_C\mathrm{K}_{D'+1}(ju_C)]\right.$$

$$\left.- u_C\sum_{j=1}^{\infty}\frac{\cosh(ju_C)}{j^{D'+\eta-2}}\mathrm{K}_{D'}(ju_C)\right\}^2 \Bigg/ \sum_{j=1}^{\infty}\frac{\cosh(ju_C)}{j^{D'+\eta-2}}\mathrm{K}_{D'}(ju_C). \tag{16}$$

Because of the general form of the external potential adopted, the expressions derived above can be used to explore the properties of the relativistic Bose gases trapped in different external potentials corresponding to different choices of the parameters t_k, ε_k and L_k.

If $t_k \to \infty$ is set, the above expressions represent the thermodynamic properties of a relativistic Bose gas confined in a D-dimensional rigid box. For example, according to equations (4) and (7), the critical temperature in this case is determined by

$$\rho\lambda_{C,nr}^D = \left(\frac{8u_C}{\pi}\right)^{1/2}\sum_{j=1}^{\infty}\frac{\sinh(ju_C)}{j^{D/2-1/2}}\mathrm{K}_{D'}(ju_C), \tag{17}$$

where $\rho = Q/V$ is the density of 'charges' and $V = \prod_{k=1}^{D}(2L_k)$ is the volume of the D-dimensional box. Equation (17) is just the same as the result obtained in [10] and coincides

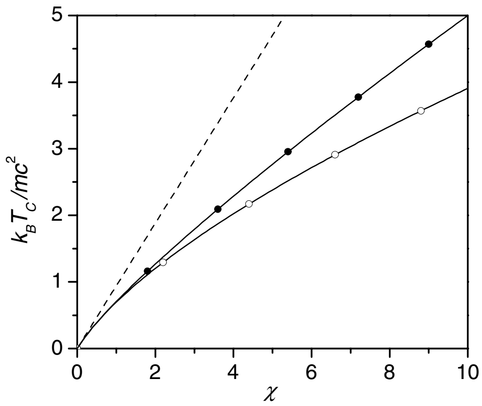

Figure 1. The scaled critical temperature k_BT_C/mc^2 as a function of the parameter χ. Solid lines with solid and empty circles represent the results of the relativistic Bose gas with and without pair production, respectively. Dashed line represents the result of the nonrelativistic approximation.

with the result given in a recently published paper [14] as long as the necessary mathematical transformation is done.

If $t_k = 2$ and $\varepsilon_k \big/ L_k^{t_k} = \gamma_k/2$ (γ_k is a positive constant) are chosen, the above expressions can be used to discuss the properties of a relativistic Bose gas trapped in a D-dimensional 'harmonic potential', which have been widely investigated under the nonrelativistic limit [18]. For example, it is found from equations (4) and (7) that the critical temperature is now determined by

$$Q\left(\frac{\hbar\varpi}{k_BT_C}\right)^D = \left(\frac{8u_C}{\pi}\right)^{1/2}\sum_{j=1}^{\infty}\frac{\sinh(ju_C)}{j^{D-1/2}}\mathrm{K}_{D'}(ju_C), \tag{18}$$

where $\varpi = \left(\prod_{k=1}^{D}\gamma_k^{1/2}\right)^{1/D}/m^{1/2}$.

3. Discussion

It is seen from equation (7) that the critical temperature is dependent on the number of 'charges' Q, the parameters of external potential t_k, ε_k and L_k and the rest mass of a particle m. If a parameter related to these quantities

$$\chi = \frac{k_BT_{C,nr}}{mc^2} = \frac{1}{mc^2}\left[\frac{Qh^D}{\zeta(\eta+D/2)(2\pi m)^{D/2}}\prod_{i=1}^{D}\frac{\varepsilon_k^{1/t_k}}{(2L_k)\Gamma(1/t_k+1)}\right]^{1/(\eta+D/2)}, \tag{19}$$

is introduced, equation (7) can be expressed as

$$\chi^{\eta+D/2} = \left(\frac{8}{\pi}\right)^{1/2}\frac{u_C^{\eta+D/2-1/2}}{\zeta(\eta+D/2)}\sum_{j=1}^{\infty}\frac{\sinh(ju_C)}{j^{D'+\eta-1}}\mathrm{K}_{D'}(ju_C), \tag{20}$$

where $T_{C,nr}$ is the nonrelativistic critical temperature of BEC [19] and $\zeta(x) = \sum_{j=1}^{\infty} 1/j^x$. Figure 1 shows the scaled critical temperature k_BT_C/mc^2 as a function of the parameter χ in the case of $D = 3$ and $\eta = 3$, which may correspond to the system trapped in a three-dmensional 'harmonic potential'. The result is compared with that obtained without considering the pair production and that obtained under the nonrelativistic approximation. It is shown that both

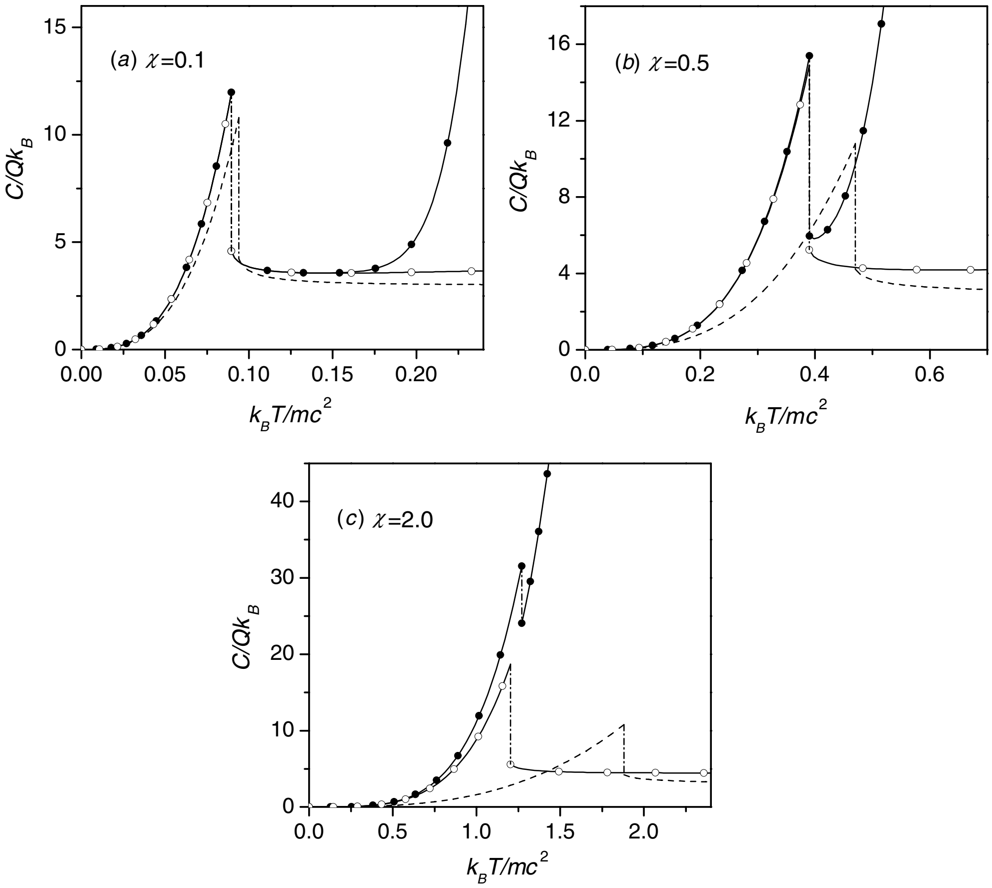

Figure 2. The scaled heat capacity C/Qk_B as a function of k_BT/mc^2 for different values of the parameter χ: (*a*) $\chi = 0.1$, (*b*) $\chi = 0.5$ and (*c*) $\chi = 2.0$. Solid lines with solid and empty circles represent the results of the relativistic Bose gas with and without pair production, respectively. The dashed line represents the results of the nonrelativistic approximation.

the relativistic effects and the influences of the pair production cannot be ignored when the parameter χ is large. It is also observed that for a large parameter χ, the pair production considerably increases the critical temperature of BEC. In particularly, in the case of $\chi \gg 1$, which implies $u_C \ll 1$, by using the approximation $\sinh(x) \xrightarrow{x\to 0} x$ and

$$\mathrm{K}_\nu(x) \xrightarrow{x\to 0} \frac{\Gamma(\nu)}{2}\left(\frac{2}{x}\right)^\nu, \tag{21}$$

one can derive from equation (20) that

$$T_C = \frac{mc^2}{k_B}\left[\frac{\pi^{1/2}\zeta(\eta + D/2)\chi^{\eta+D/2}}{2^{D/2+1}\Gamma(D/2 + 1/2)\zeta(\eta + D - 1)}\right]^{1/(\eta+D-1)}. \tag{22}$$

Equation (22) shows that BEC may occur at a very high critical temperature ($k_BT_C \gg mc^2$) for the system with $\chi \gg 1$. The analogous phenomenon has been studied for an ideal Bose gas confined in a rigid box [7, 10, 13, 14], where the parameter χ is reduced to

$$\chi = \frac{(\rho^{1/D}\lambda_c)^2}{2\pi[\zeta(D/2)]^{2/D}}, \tag{23}$$

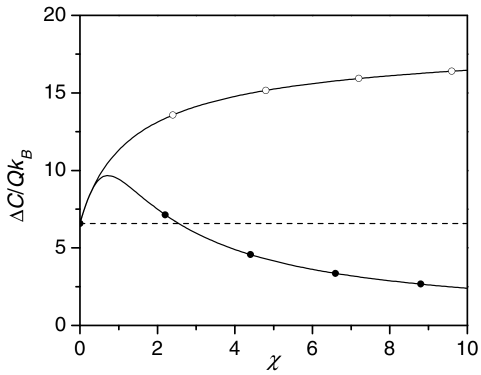

Figure 3. The jump of the heat capacity at the critical point $\Delta C/Qk_B$ as a function of the parameter χ. Solid lines with solid and empty circles represent the results of the relativistic Bose gases with and without pair production, respectively. The dashed line represents the result of the nonrelativistic approximation.

and $\lambda_c = h/mc$ is the 'Compton wave length'. The condition $\chi \gg 1$ can now be written as $\rho^{1/D}\lambda_c \gg 1$, which corresponds to a system of high density of particles and/or small rest mass of a particle. The power-law potential adopted here plays a role similar to the rigid box confining the Bose gas and the strong potential, which will lead to the high density of particles around the center of the potential, corresponds to the small volume of the box.

Using equations (3), (13) and (15), one can explore the dependence of the heat capacity on the temperature, as shown in figure 2, where (*a*), (*b*) and (*c*) correspond to the cases of $\chi = 0.1, 0.5$ and 2.0, respectively. It is interesting to note that the pair production greatly alters the high-temperature behavior of the heat capacity. For example, the heat capacity comes close to a constant at high temperatures for the Bose gas without pair production, while it increases rapidly at high temperatures for the system with pair production. The result can be explained as follows: according to equation (10), the contributions of particles and antiparticles to the total energy are both positive. The production of large numbers of particle–antiparticle pairs at high temperatures will significantly increase the total energy, and hence increase the heat capacity of the system. Comparing the curves for $\chi = 0.1$, 0.5 and 2.0, one can find that the dependence of the heat capacity on the temperature is obviously different for the different parameters χ when the pair production is taken into account. In the case of a small χ (e.g. $\chi = 0.1$ and 0.5), $C_{T>T_C}$ is not a monotonic function of temperature. It first decreases with temperature and reaches a minimal value. It rapidly increases with temperature at high temperatures. In the case of a large χ (e.g. $\chi = 2.0$), however, $C_{T>T_C}$ increases monotonously with temperatures in the entire region of $T > T_C$.

Figure 3 gives the curves of the jump of the heat capacity varying with the parameter χ. It is found that the pair production significantly reduces the gap of the heat capacity at the critical temperature for the systems with a large parameter χ. In the case of $\chi \gg 1$, i.e., $u_C \ll 1$, by using $\sinh(x) \xrightarrow{x\to 0} x$, $\cosh(x) \xrightarrow{x\to 0} 1$ and equation (21), equation (16) can be expressed as

$$\Delta C = C_{T\to T_C^-} - C_{T\to T_C^+} = Qk_B(\eta + D - 1)^2 u_C. \tag{24}$$

Equation (24) shows that $\Delta C \to 0$ when $\chi \gg 1$, i.e., $u_C \ll 1$. This indicates that the anomaly of the heat capacity at the critical temperature is completely removed in the ultrarelativistic limit when the particles–antiparticle pair production is taken into account.

4. Conclusions

We have studied the properties of a relativistic Bose gas in an external potential, in which the effects of particle–antiparticle pair production are taken into account. Some important results have been obtained as follows. (1) The relativistic effects on the characteristics of BEC are considerable for the Bose system with a large parameter χ. (2) The pair production increases the critical temperature and BEC may occur at high temperatures ($k_B T_C \gg mc^2$) for the system with the parameter $\chi \gg 1$. (3) The pair production significantly reduces the jump of the heat capacity at the critical temperature and the anomaly of the heat capacity at the critical temperature is completely removed in the ultrarelativistic limit. (4) The pair production results in the rapid increase of the heat capacity at high temperatures.

Because of the relativistic energy dispersion and general form of the external potential adopted, the results obtained in the present paper may include many significant results in the literature.

Acknowledgment

This work has been supported by the Research Foundation of Ministry of Education (no. 20050384005), People's Republic of China.

References

[1] Ayala A and Smerzi A 1997 *Phys. Lett.* B **405** 20
[2] Begun V V and Gorenstein M I 2006 *Phys. Rev.* C **73** 054904
[3] Landsberg P T and Dunning-Davies J 1965 *Phys. Rev.* A **138** 1049
[4] Araglo de Carvalho C A and Goulart Rosa S Jr 1980 *J. Phys. A: Math. Gen.* **13** 3233
[5] Beckmann R, Karsch F and Miller D E 1979 *Phys. Rev. Lett.* **43** 1277
Beckmann R, Karsch F and Miller D E 1982 *Phys. Rev.* A **25** 561
[6] Dixit V V and Miller D E 1982 *Phys. Rev.* A **28** 1709
[7] Haber H E and Weldon H A 1981 *Phys. Rev. Lett.* **46** 1497
Haber H E and Weldon H A 1982 *Phys. Rev.* D **25** 502
[8] Singh S and Pathria R K 1984 *Phys. Rev.* A **30** 442, 3198
[9] Shiokawa K and Hu B L 1999 *Phys. Rev.* D **60** 105016
[10] Frota H O, Silva M H and Goulart Rosa S Jr 1989 *Phys. Rev.* A **39** 830
[11] Toms D J 1992 *Phys. Rev. Lett.* **69** 1152
Toms D J 1993 *Phys. Rev.* D **47** 2483
[12] Burakovsky L, Horwitz L P and Schieve W C 1996 *Phys. Rev.* D **54** 4029
[13] Salasnich L 2002 *Nuovo Cimento* B **117** 637
[14] Grether M, de Llano M and Baker G A Jr 2007 *Phys. Rev. Lett.* **99** 200406
[15] Su G, Chen J and Chen L 2006 *J. Phys. A: Math. Gen.* **39** 4935
[16] Huang K 1987 *Statistical Mechanics* (New York: Wiley)
[17] Yan Z 2000 *Phys. Rev.* A **61** 063607
[18] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 *Rev. Mod. Phys.* **71** 463
[19] Yan Z 1999 *Phys. Rev.* A **59** 4657